

# Lambda-Perturbations of Keplerian Orbits

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To estimate the influence of the dark energy on the planetary orbits, we solve the general relativistic equations of motion of a test particle in the field of a point-like mass embedded in the cosmological background formed by the Lambda-term. It is found that under certain relations between three crucial parameters of the problem—the initial radius of the orbit, Schwarzschild and de Sitter radii—a secular perturbation caused by the Lambda-term becomes significant, *i.e.* can reach the rate of the standard Hubble flow.

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The question if the planetary orbits and other local celestial objects are subject to the cosmological influences (in particular, if they feel the universal Hubble expansion) was put forward by McVittie as early as 1933 [1]; and this problem attracted attention of a number of other researchers during the few subsequent decades [2–12]. A quite comprehensive review was given by Bonnor [13].

The most frequent conclusion of these studies was that the effect of cosmological expansion at the planetary scales should be very small or absent at all. However, the particular estimates given by different authors substantially disagree with each other. Moreover, the most of these estimates (excluding the recent ones) are not applicable to the case when the cosmological background is formed by the “dark energy” (*i.e.* the  $\Lambda$ -term in Einstein equations), just because it is distributed perfectly uniform and insensitive to the local gravitational perturbations.

For example, the most well-known argument against the local Hubble expansion is the so-called Einstein–Straus theorem [2]: Let us consider a uniform background cosmological matter distribution and, next, cut out a spherical cavity and concentrate all its mass in the central point. Then, a solution of the General Relativity equations will be given by the purely static Schwarzschild metric inside the cavity, and it will transform to the time-dependent Friedmann–Robertson–Walker metric outside the cavity. In other words, there is no Hubble expansion in the local empty neighborhood of the point-like massive body, but such an expansion appears in the regions of space filled with the cosmological background matter. Unfortunately, despite an apparent generality of this result, it is evidently inapplicable to the  $\Lambda$ -dominated cosmology, because it is meaningless to consider an empty cavity in the vacuum energy distribution. Similarly, it can be shown that such arguments against the local Hubble expansion as the “virial criterion” of gravitational binding and Einstein–Infeld–Hoffmann surface integral method [4] also do not work when the cosmological background is formed by the perfectly uniform  $\Lambda$ -term.

Therefore, the most straightforward and self-consistent way to estimate how much can the dark energy affect the

planetary dynamics is just to solve the two-body problem in the pure  $\Lambda$ -background. In the simplest case of a test particle of infinitely small mass moving around the point-like mass  $M$ , this can be done using the well-known solution of General Relativity equations obtained long time ago by Kottler [14] (in the modern literature, it is often called Schwarzschild–de Sitter solution):

$$ds^2 = - \left( 1 - \frac{2GM}{c^2 r'} - \frac{\Lambda r'^2}{3} \right) c^2 dt'^2 + \left( 1 - \frac{2GM}{c^2 r'} - \frac{\Lambda r'^2}{3} \right)^{-1} dr'^2 + r'^2 (d\theta^2 + \sin^2 \theta d\varphi^2); \quad (1)$$

for a more general review, see also [15]. Here,  $G$  is the gravitational constant,  $c$  is the speed of light, and primes designate the original Kottler’s “static” coordinates.

Since metric (1) was derived well before the birth of the modern cosmology, it suffers from a lack of the adequate cosmological asymptotics at infinity; namely, it does not reproduce the standard Hubble flow. Unfortunately, this fact was ignored in a number of recent studies [24]. To resolve the above problem, it is necessary to perform transformation to the commonly-used cosmological Robertson–Walker coordinates (represented below by the variables without primes):

$$r' = a_0 \exp\left(\frac{ct}{r_\Lambda}\right) r, \quad (2a)$$

$$t' = t - \frac{1}{2} \frac{r_\Lambda}{c} \ln \left[ 1 - \frac{a_0^2}{r_\Lambda^2} \exp\left(\frac{2ct}{r_\Lambda}\right) r^2 \right], \quad (2b)$$

as outlined in our earlier work [16]. As a result, the metric will take the form:

$$ds^2 = g_{tt} c^2 dt^2 + 2 g_{tr} c dt dr + g_{rr} dr^2 + g_{\theta\theta} d\theta^2 + g_{\varphi\varphi} d\varphi^2, \quad (3)$$

where

$$g_{tt} = \frac{- \left( 1 - \frac{r_g}{r'} - \frac{r'^2}{r_\Lambda^2} \right)^2 + \left( 1 - \frac{r'^2}{r_\Lambda^2} \right)^2 \frac{r'^2}{r_\Lambda^2}}{\left( 1 - \frac{r_g}{r'} - \frac{r'^2}{r_\Lambda^2} \right) \left( 1 - \frac{r'^2}{r_\Lambda^2} \right)}, \quad (4a)$$

$$g_{tr} = \frac{\left(1 - \frac{r'^2}{r_\Lambda^2}\right)^2 - \left(1 - \frac{r_g}{r'} - \frac{r'^2}{r_\Lambda^2}\right)^2}{\left(1 - \frac{r_g}{r'} - \frac{r'^2}{r_\Lambda^2}\right)\left(1 - \frac{r'^2}{r_\Lambda^2}\right)^2} \frac{r'}{r_\Lambda} \frac{r'}{r}, \quad (4b)$$

$$g_{rr} = \frac{\left(1 - \frac{r'^2}{r_\Lambda^2}\right)^2 - \left(1 - \frac{r_g}{r'} - \frac{r'^2}{r_\Lambda^2}\right)^2 \frac{r'^2}{r_\Lambda^2}}{\left(1 - \frac{r_g}{r'} - \frac{r'^2}{r_\Lambda^2}\right)\left(1 - \frac{r'^2}{r_\Lambda^2}\right)^2} \frac{r'^2}{r^2}, \quad (4c)$$

$$g_{\theta\theta} = g_{\varphi\varphi}/\sin^2\theta = r'^2. \quad (4d)$$

In the above formulas,  $r_g = 2GM/c^2$  is Schwarzschild radius,  $r_\Lambda = \sqrt{3/\Lambda}$  is de Sitter radius, and  $a_0$  is the scale factor of the Universe.

Taking  $a_0 = 1$  at  $t = 0$  and keeping only the lowest-order terms of  $r_g$  and  $1/r_\Lambda$ , metric (4a)–(4d) can be rewritten in a more compact form:

$$g_{tt} \approx -\left[1 - \frac{2GM}{c^2 r} \left(1 - \frac{c\sqrt{\Lambda}t}{\sqrt{3}}\right)\right], \quad (5a)$$

$$g_{tr} \approx \frac{4GM\sqrt{\Lambda}}{\sqrt{3}c^2}, \quad (5b)$$

$$g_{rr} \approx \left[1 + \frac{2GM}{c^2 r} \left(1 - \frac{c\sqrt{\Lambda}t}{\sqrt{3}}\right)\right] \left(1 + \frac{2c\sqrt{\Lambda}t}{\sqrt{3}}\right), \quad (5c)$$

$$g_{\theta\theta} = g_{\varphi\varphi}/\sin^2\theta \approx r^2 \left(1 + \frac{2c\sqrt{\Lambda}t}{\sqrt{3}}\right). \quad (5d)$$

Motion of a test particle in this metric is described by the geodesic equations which can be derived by the standard way:

$$\begin{aligned} 2 \left[1 - \frac{r_g}{r} \left(1 - \frac{t}{r_\Lambda}\right)\right] \ddot{t} - 4 \frac{r_g}{r_\Lambda} \ddot{r} + \frac{r_g}{r_\Lambda} \frac{1}{r} \dot{t}^2 \\ + 2 \frac{r_g}{r^2} \left(1 - \frac{t}{r_\Lambda}\right) \dot{t} \dot{r} + \frac{1}{r_\Lambda} \left(2 + \frac{r_g}{r}\right) \dot{r}^2 \\ + 2 \frac{r^2}{r_\Lambda} \left(\dot{\theta}^2 + \sin^2\theta \dot{\varphi}^2\right) = 0, \end{aligned} \quad (6a)$$

$$\begin{aligned} 4 \frac{r_g}{r_\Lambda} \ddot{t} + 2 \left[1 + 2 \frac{t}{r_\Lambda} + \frac{r_g}{r} \left(1 + \frac{t}{r_\Lambda}\right)\right] \ddot{r} \\ + \frac{r_g}{r^2} \left(1 - \frac{t}{r_\Lambda}\right) \dot{t}^2 + \frac{2}{r_\Lambda} \left(2 + \frac{r_g}{r}\right) \dot{t} \dot{r} - \frac{r_g}{r^2} \left(1 + \frac{t}{r_\Lambda}\right) \dot{r}^2 \\ - 2r \left(1 + 2 \frac{t}{r_\Lambda}\right) \left(\dot{\theta}^2 + \sin^2\theta \dot{\varphi}^2\right) = 0, \end{aligned} \quad (6b)$$

$$\begin{aligned} r \left(1 + 2 \frac{t}{r_\Lambda}\right) \ddot{\theta} + 2 \frac{r}{r_\Lambda} \dot{t} \dot{\theta} + 2 \left(1 + 2 \frac{t}{r_\Lambda}\right) \dot{r} \dot{\theta} \\ - r \left(1 + 2 \frac{t}{r_\Lambda}\right) \sin\theta \cos\theta \dot{\varphi}^2 = 0, \end{aligned} \quad (6c)$$

$$\begin{aligned} r \left(1 + 2 \frac{t}{r_\Lambda}\right) \sin\theta \ddot{\varphi} + 2 \frac{r}{r_\Lambda} \sin\theta \dot{t} \dot{\varphi} + 2 \left(1 + 2 \frac{t}{r_\Lambda}\right) \sin\theta \dot{r} \dot{\varphi} \\ + 2r \left(1 + 2 \frac{t}{r_\Lambda}\right) \cos\theta \dot{\theta} \dot{\varphi} = 0, \end{aligned} \quad (6d)$$

where, for conciseness, we put  $c \equiv 1$ , and dot denotes a derivative with respect to the proper time of the moving particle.

Next, if the coordinate system is oriented so that the particle moves in the equatorial plane,  $\theta = \pi/2 = \text{const}$ , these equations are reduced to the following set:

$$\begin{aligned} 2 \left[1 - \frac{r_g}{r} \left(1 - \frac{t}{r_\Lambda}\right)\right] \ddot{t} - 4 \frac{r_g}{r_\Lambda} \ddot{r} + \frac{r_g}{r_\Lambda} \frac{1}{r} \dot{t}^2 \\ + 2 \frac{r_g}{r^2} \left(1 - \frac{t}{r_\Lambda}\right) \dot{t} \dot{r} + \frac{1}{r_\Lambda} \left(2 + \frac{r_g}{r}\right) \dot{r}^2 \\ + 2 \frac{r^2}{r_\Lambda} \dot{\varphi}^2 = 0, \end{aligned} \quad (7a)$$

$$\begin{aligned} 4 \frac{r_g}{r_\Lambda} \ddot{t} + 2 \left[1 + 2 \frac{t}{r_\Lambda} + \frac{r_g}{r} \left(1 + \frac{t}{r_\Lambda}\right)\right] \ddot{r} \\ + \frac{r_g}{r^2} \left(1 - \frac{t}{r_\Lambda}\right) \dot{t}^2 + \frac{2}{r_\Lambda} \left(2 + \frac{r_g}{r}\right) \dot{t} \dot{r} \\ - \frac{r_g}{r^2} \left(1 + \frac{t}{r_\Lambda}\right) \dot{r}^2 - 2r \left(1 + 2 \frac{t}{r_\Lambda}\right) \dot{\varphi}^2 = 0, \end{aligned} \quad (7b)$$

$$\begin{aligned} r \left(1 + 2 \frac{t}{r_\Lambda}\right) \ddot{\varphi} + 2 \frac{r}{r_\Lambda} \dot{t} \dot{\varphi} \\ + 2 \left(1 + 2 \frac{t}{r_\Lambda}\right) \dot{r} \dot{\varphi} = 0. \end{aligned} \quad (7c)$$

In the case of a realistic planetary system, the problem under consideration involves three characteristic scales, which differ from each other by many orders of magnitude—Schwarzschild radius  $r_g$  (e.g.,  $\sim 10^{-2}$  m for the Earth as a central body), a typical initial radius of the planetary orbit  $R_0$  (e.g.,  $\sim 10^9$  m for the Moon moving around the Earth), and de Sitter radius  $r_\Lambda$  ( $\sim 10^{27}$  m, which depends on the amount of dark energy in the Universe) [25].

Despite of the availability of the small ratios, the analytical treatment of set of equations (7a)–(7c) is very hard, because the corresponding expansions in terms of the small parameters do not converge. The numerical integration is also challenging, because the standard accuracy of representation of the floating-point numbers in a computer is usually insufficient to cover the above-mentioned range of numbers. As a result, it is necessary to use a special software for emulation of the high-accuracy arithmetic.

A detailed numerical analysis of these equations will be published elsewhere; while here we restrict our consideration by a few toy models, where difference between the characteristic scales is not so much as in reality. This will help us to reveal the most important features of the resulting motion. From here on, it is convenient to take the initial radius of the orbit  $R_0$  as a unit of length. Then, the unit of time will be  $cR_0$  (or just  $R_0$  if  $c \equiv 1$ , as in the previous equations). The corresponding dimensionless variables, normalized by  $R_0$  and  $cR_0$ , will be denoted by asterisks.

So, let us take, for example,  $r_g^* = 0.01$ ,  $R_0^* = 1$  and study the characteristic shapes of the test-particle orbits

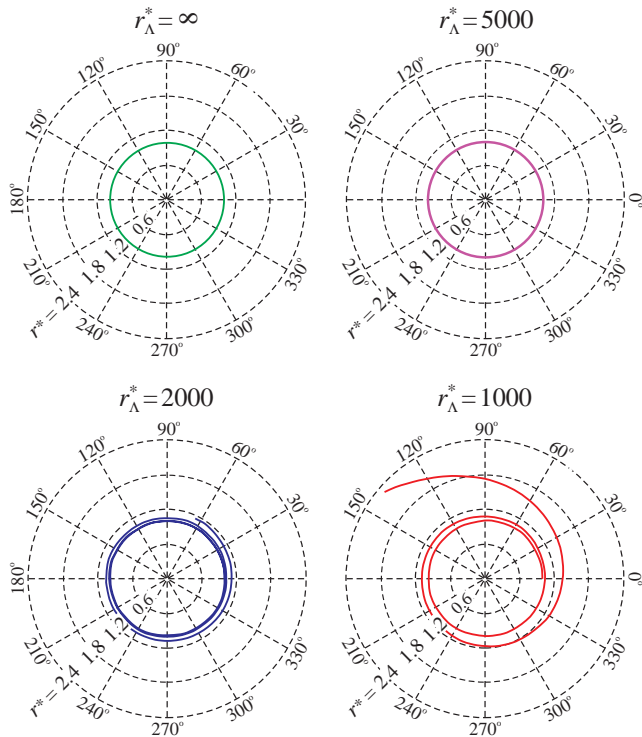


FIG. 1: Orbits of the test particles at the specified Schwarzschild radius  $r_g^* = 0.01$  and various de Sitter radii  $r_\Lambda^*$  (*i.e.*, various values of the  $\Lambda$ -term).

at various values of  $r_\Lambda^*$ . Results of numerical integration of the equations of motion (7a)–(7c) for a few values of de Sitter radius are shown in Fig. 1. If  $r_\Lambda^* = \infty$  (*i.e.*,  $\Lambda = 0$ ), the orbit is closed, as should be evidently expected. Next, when the values of  $r_\Lambda^*$  decrease down to a few thousand (*i.e.*, the values of  $\Lambda$  increase), the orbits become slightly spiral; and such an unwinding is expressed very well, for example, at  $r_\Lambda^* = 1000$ .

The same orbital radii  $r^*$  are presented as functions of the coordinate time  $t^*$  (which differs, in fact, only slightly from the proper time of the moving particle) in Fig. 2. The curves are wavy because the initial unperturbed orbit was taken to be slightly elliptic. The straight lines in this figure represent the standard Hubble motion experienced by the test particle (*i.e.*, without any central gravitating body) at the same values of  $r_\Lambda^*$ .

It is evident that under certain circumstances (depending on the ratio between the above-mentioned characteristic parameters) the perturbation of the planetary orbit by the dark energy can be appreciable, and the rate of increase in the orbital radius can be even comparable to the rate of the standard Hubble flow at infinity. In our opinion, this points to the potential importance of the local cosmological influences on the planetary dynamics, although a much more careful analysis (with realistic planetary parameters and the additional factors affecting the planetary dynamics) is still to be done.

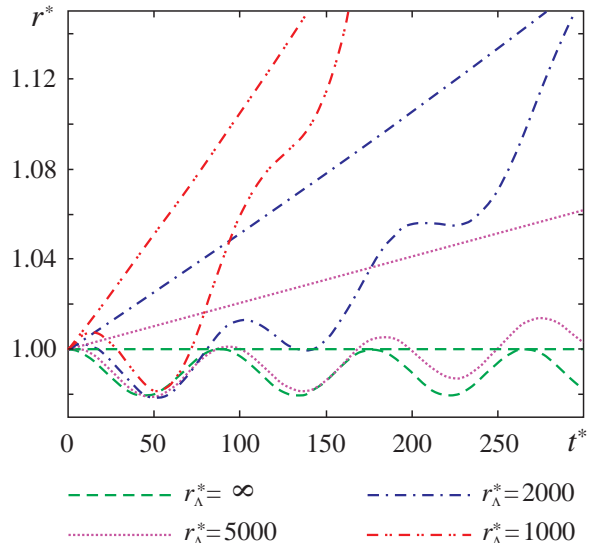


FIG. 2: Radii of the orbits as functions of time at the specified Schwarzschild radius of the central body  $r_g^* = 0.01$  and various de Sitter radii  $r_\Lambda^*$  (wavy curves) vs. radii of the test particles in the standard Hubble flow (straight lines).

It should be mentioned also that there is some empirical evidence in favor of the probable cosmological expansion in the Earth–Moon system: This is a well-known disagreement between the rates of secular increase in the lunar semi-major axis measured, on the one hand, immediately by the lunar laser ranging [17],  $\dot{R}_{\text{LLR}} = 3.8 \pm 0.1$  cm/yr [18], and, on the other hand, derived indirectly from the data on the Earth’s rotation deceleration,  $\dot{R}_{\text{rot}} = 1.6 \pm 0.2$  cm/yr [*e.g.*, the time series compiled in 19]. It is quite surprising that these two values can be reconciled with each other quite well if, along with the commonly-considered tidal interaction between the Earth and Moon, one takes into account also a contribution from the local Hubble expansion; and the rate of such an expansion turns out to be in reasonable agreement with the large-scale cosmological data [20].

On the other hand, it is commonly believed that the modern solar-system ephemerides are able to explain all the planetary motions without any additional cosmological influences, apart from the well-known post-Newtonian corrections (see, for example, papers [21, 22] and references therein). If this is really the case, and no further corrections for the  $\Lambda$ -term are necessary, then the high-accuracy planetary observations can be used to impose strong constraints on the amount of the dark energy represented by the perfectly uniform and constant  $\Lambda$ -term. So, the models with the “dynamic” dark energy (scalar field) and/or the nontrivial equation of state may become preferable.

In summary, we presented a rigorous mathematical formulation of the two-body problem in the  $\Lambda$ -dominated cosmology with an adequate asymptotics at infinity. A

set of solutions of the respective equations of motion was obtained numerically. It was found that, in some circumstances, the secular perturbation of orbits by the  $\Lambda$ -term can be appreciable and even reach the rate of the standard Hubble flow. This fact implies that a possible presence of the local cosmological influences should be taken into account very carefully in the future analyses of dynamics of the planetary and other binary systems.

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- [24] Therefore, such works analyzed only the “conservative” effects caused by the  $\Lambda$ -term; while the cosmological influences, as such, were ignored *a priori*.
- [25] It was emphasized for the first time by Balaguera-Antolínez et al. [9] that the specific interplay between  $r_g$  and  $r_\Lambda$  can result in the manifestation of  $\Lambda$ -term effects at the spatial scales much less than  $r_\Lambda$ ; but that consideration was performed for the static Kottler metric (1). Besides, the “small-scale” cosmological effects were found also in the collapsing matter overdensities (*e.g.*, paper [23] and references therein); but such analyses were performed in the models of “dynamical” dark energy and are, therefore, irrelevant to our study.